Integral Solutions - Integral Three
Barrier Probabilities

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We want to find the probability that a Brownian motion that starts out at a value of zero at time zero, falls below some minimum barrier value y over the time interval $[0, t]$, and ends up above some threshold value x at time t given the following constraints...

$$\text{Constraints}: \;\; \text{barrier } y < 0 \;\; \text{...and...} \;\; \text{threshold } x \geq y \tag{1}$$

Setting Up The Problem

We will define the variables μ to be expected return mean, the variable ϕ to be the dividend yield, and the variable σ to be expected return volatility. We will define the variables α and υ to be the mean and variance, respectively, of the Brownian motion over the time interval $[0, T]$. The equations for the Brownian motion's mean and variance are...

$$\alpha = \left(\mu - \phi - \frac{1}{2}\sigma^2\right)t \;\; \text{...and...} \;\; \upsilon = \sigma^2 t \tag{2}$$

We will define the variable m to be the minimum value of the Brownian motion W_t over the time interval $[0, t]$ and the variable w to be the ending value of the Brownian motion at time t. We will define the function $a(m, w)$ to be the joint distribution function of m and w, which is...

$$a(m, w) = \frac{2\,(w - 2m)}{\upsilon\sqrt{2\pi\upsilon}} \,\text{Exp}\left\{\frac{\alpha w}{\upsilon} - \frac{\alpha^2}{2\upsilon} - \frac{1}{2\upsilon}(2m - w)^2\right\} \tag{3}$$

We will define the variable x to be the threshold value and the variable y to be the barrier value. Using Equation (3) above the equation for the probability that (1) the minimum value of W_t is less than or equal to the barrier value over the time interval $[0, T]$ and (2) the ending value of W_t is greater than or equal to the threshold value at time T is...

$$\text{Prob}\Big[\text{Min}(W_t) \leq y, W_T \geq x\Big] = \int_{w=x}^{w=\infty}\int_{m=-\infty}^{m=y} a(m, w)\,\delta m\,\delta w \tag{4}$$

To solve the double integral in Equation (4) abov we need the anti-derivative of the function $a(m, w)$ with respect to the barrier variable m. The antiderivative of the function $a(m, w)$ is $b(m, w)$, which is defined as follows... [1]

$$b(m, w) = \sqrt{\frac{1}{2\pi\upsilon}} \,\text{Exp}\left\{\frac{\alpha w}{\upsilon} - \frac{\alpha^2}{2\upsilon} - \frac{1}{2\upsilon}(2m - w)^2\right\} \;\text{...such that...}\; \frac{\delta\, b(m, w)}{\delta m} = a(m, w) \tag{5}$$

Solution To The Inner Integral

We will define the variable I to be the inner integral of the double integral as defined by Equation (4) above. This statement in equation form is...

$$I = \int\limits_{m=-\infty}^{m=y} a(m, w)\,\delta m \tag{6}$$

Using Equation (5) above the solution to Equation (6) above is...

$$I = \int\limits_{m=-\infty}^{m=y} a(m, w)\,\delta m = b(m, w)\Bigg[_{m=-\infty}^{m=y} \tag{7}$$

Note that using Equation (5) above we can rewrite Equation (7) above as...

$$I = \frac{1}{\sqrt{2\pi\upsilon}} \text{Exp}\left\{\frac{\alpha w}{\upsilon} - \frac{\alpha^2}{2\upsilon} - \frac{1}{2\upsilon}(2m - w)^2\right\}\Bigg[_{m=-\infty}^{m=y} \tag{8}$$

The solution to the lower bound of the integral in Equation (8) above is...

$$\sqrt{\frac{1}{2\pi\upsilon}} \text{Exp}\left\{\frac{\alpha w}{\upsilon} - \frac{\alpha^2}{2\upsilon} - \frac{1}{2\upsilon}(2 \times -\infty - w)^2\right\} = 0 \text{ ...because... } \lim_{m\to-\infty} \text{Exp}\left\{m\right\} = 0 \tag{9}$$

The solution to the upper bound of the integral in Equation (8) above is...

$$\sqrt{\frac{1}{2\pi\upsilon}} \text{Exp}\left\{\frac{\alpha w}{\upsilon} - \frac{\alpha^2}{2\upsilon} - \frac{1}{2\upsilon}(2 \times y - w)^2\right\} \tag{10}$$

Using Equations (9) and (10) above the solution to Equation (8) above is...

$$\begin{aligned} I &= \sqrt{\frac{1}{2\pi\upsilon}} \text{Exp}\left\{\frac{\alpha w}{\upsilon} - \frac{\alpha^2}{2\upsilon} - \frac{1}{2\upsilon}(2\,y - w)^2\right\} - 0 \\ &= \sqrt{\frac{1}{2\pi\upsilon}} \text{Exp}\left\{\frac{\alpha w}{\upsilon} - \frac{\alpha^2}{2\upsilon} - \frac{1}{2\upsilon}(4\,y^2 - 4\,w\,y + w^2)\right\} \\ &= \sqrt{\frac{1}{2\pi\upsilon}} \text{Exp}\left\{-\frac{1}{2\upsilon}\,(-2\,\alpha\,w + \alpha^2 + 4\,y^2 - 4\,w\,y + w^2)\right\} \end{aligned} \tag{11}$$

We will make the following definition...

$$(w - \alpha - 2\,y)^2 = w^2 - 2\,\alpha\,w + \alpha^2 + 4\,y^2 - 4\,w\,y + 4\,\alpha\,y \tag{12}$$

Using the definition in Equation (12) above we can rewrite Equation (11) above as...

$$\begin{aligned} I &= \sqrt{\frac{1}{2\pi\upsilon}} \text{Exp}\left\{-\frac{1}{2\upsilon}\,((w - \alpha - 2\,y)^2 - 4\,\alpha\,y)\right\} \\ &= \text{Exp}\left\{\frac{2\,\alpha\,y}{\upsilon}\right\} \sqrt{\frac{1}{2\pi\upsilon}} \text{Exp}\left\{-\frac{1}{2\upsilon}\,(w - \alpha - 2\,y)^2\right\} \end{aligned} \tag{13}$$

Solution To The Outer Integral

Using Equation (13) above we can rewrite probability Equation (4) above as...

$$\text{Prob}\Big[\text{Min}(W_t) \le y, W_T \ge x\Big] = \int_{w=x}^{w=\infty} I\,\delta w = \int_{w=x}^{w=\infty} \text{Exp}\left\{\frac{2\,\alpha\,y}{\upsilon}\right\} \sqrt{\frac{1}{2\pi\upsilon}} \text{Exp}\left\{-\frac{1}{2\upsilon}\,(w - \alpha - 2\,y)^2\right\} \delta w \tag{14}$$

Not that we can rewrite Equation (14) above as...

$$\text{Prob}\Big[\text{Min}(W_t) \le y, W_T \ge x\Big] = \text{Exp}\left\{\frac{2\,\alpha\,y}{\upsilon}\right\} \int_{w=x}^{w=\infty} \sqrt{\frac{1}{2\pi\upsilon}} \text{Exp}\left\{-\frac{1}{2\upsilon}\,(w - (\alpha + 2\,y))^2\right\} \delta w \tag{15}$$

We will define the function CNDF(z) to be the cumulative normal distribution function were the random variable z is normally-distributed with mean = [*mean*] and variance = [*variance*]. The equation for this function is...

$$CNDF(x, mean, variance) = \int_{z=-\infty}^{z=x} \sqrt{\frac{1}{2\,\pi} \times \frac{1}{variance}} \times \text{Exp}\left\{-\frac{1}{2} \times \frac{1}{variance} \times (z - mean)^2\right\} \delta z \tag{16}$$

The Excel function for Equation (16) above is...

$$CNDF(x, mean, variance) = NORMDIST(x, mean, \sqrt{variance}, True) \tag{17}$$

Using the function definition in Equation (16) above the solution to Equation (15) above is...

$$\text{Prob}\Big[\text{Min}(W_t) \le y, W_T \ge x\Big] = \text{Exp}\left\{\frac{2\,\alpha\,y}{\upsilon}\right\} \left(1 - CNDF\Big[x, \alpha + 2\,y, \upsilon\Big]\right) \tag{18}$$

Example

Given the model assumptions below what is the probability that a Brownian motion that starts out at a value of zero at time zero, falls below some minimum barrier value y over the time interval $[0, T]$, and ends up above some threshold value x at time T?

Table 1: Model Assumptions

Symbol	Description	Value
y	Barrier value	-0.50
x	Threshold value	-0.25
μ	Annual expected return - mean (%)	6.00
σ	Annual expected return - Volatility (%)	35.00
ϕ	Annual dividend yield (%)	2.50
T	Time period in years (#)	5.00

Using Equation (2) above the equations for the Brownian motion's mean and variance are...

$$\alpha = \left(0.0600 - 0.0250 - \frac{1}{2} \times 0.3500^2\right) \times 5 = -0.13125 \text{ ...and... } \upsilon = 0.3500^2 \times 5 = 0.61250 \tag{19}$$

Using Equations (17), (18) and (19) above the solution to our problem is...

$$\text{Answer} = \text{Exp}\left\{\frac{2 \times -0.13125 \times -0.50}{0.61250}\right\} \times (1 - NORMDIST(-0.25, -0.13125 + 2 \times -0.50, \sqrt{0.61250}, True)) = 0.16116 \tag{20}$$

References

[1] Integral Solutions - Integral Two (Joint Distribution Anti-Derivative), Schurman